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Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Heat Conduction Across a Sandwiched Plate with Stringers

C. Y. Wang*
Michigan State University,
East Lansing, Michigan 48824

I. Introduction

THE sandwiched plate has long been an important structural element in the aerospace industry. Figure 1a shows a typical cross section of a sandwich plate. The two thin facing sheets are separated by a light weight core with evenly spaced stringers. The aim of this Note is to study the local heat conduction through this sandwiched plate.

The facing sheets usually have very little thermal resistivity and can be ignored in the analysis.[1] The core is composed of two different (each assumed homogeneous) materials periodically spaced. If the surfaces of this composite plate are at different uniform temperatures, the temperature distribution would be everywhere linear across the plate. The situation is much more complicated for the case when one side is heated by a constant flux. This interesting case is studied in this Note.

II. Formulation

Let the thickness of the plate be H. The plate has constant temperature T_0 on the top surface and constant heat flux q on the bottom surface. Let the temperature and the thermal conductivity be T_i' and k_i, respectively, where $i = 1, 2$ indicates the two different homogeneous materials. We normalize all lengths by H, the temperature by

$$T_i' = T_0 + \frac{qH}{k_1} T_i, \qquad i = 1, 2 \tag{1}$$

Figure 1b shows two sets of Cartesian axes located at the bottom center of each region. The governing equation for both regions is

$$\frac{\partial^2 T_i}{\partial x_i^2} + \frac{\partial^2 T_i}{\partial y^2} = 0 \tag{2}$$

with the boundary conditions

$$T_i = 0 \quad \text{on} \quad y = 1 \tag{3}$$

$$\frac{\partial T_1}{\partial y} = -1 \quad \text{on} \quad y = 0, \qquad -a \le x_1 \le a \tag{4}$$

$$\frac{\partial T_2}{\partial y} = -\lambda \quad \text{on} \quad y = 0, \qquad -b \le x_2 \le b \tag{5}$$

Here, $\lambda \equiv k_1/k_2$ is the ratio of conductivities. Regions 1 and 2 are coupled by the transfer of heat across their common boundary. The two independent conditions are

$$\lambda \frac{\partial T_1}{\partial x}(a, y) = \frac{\partial T_2}{\partial x}(-b, y) = \frac{1}{\beta}[T_2(-b, y) - T_1(a, y)] \tag{6}$$

Here, $\beta \equiv k_2/hH$ and h is the interface heat transfer coefficient.[1] If the materials of regions 1 and 2 have perfect thermal contact, β is zero and the temperatures across the interface are equal. If there is no thermal contact, β is infinity and the interface would be an adiabatic barrier.

Equations (2–6) are to be solved for given a, b, λ, and β. Instead of using numerical methods such as finite differences, we shall use the eigenfunction expansions method which is much simpler for this particular problem.

III. Solution

Note the following series satisfy Eqs. (2–5):

$$T_1(x_1, y) = 1 - y + \sum_{n=1}^{\infty} A_n \cos(\alpha_n y)[e^{-\alpha_n(x_1+a)} + e^{\alpha_n(x_1-a)}] \tag{7}$$

$$T_2(x_2, y) = \lambda(1 - y) + \sum_{n=1}^{\infty} B_n \cos(\alpha_n y)[e^{-\alpha_n(x_2+b)} + e^{\alpha_n(x_2-b)}] \tag{8}$$

where $\alpha_n \equiv (n - \frac{1}{2})\pi$ and A_n, B_n are constant coefficients. Equations (7) and (8) also satisfy the conditions of symmetry with respect to x_1, x_2. The first part of Eq. (6) yields

$$\lambda \sum_{n=1}^{\infty} A_n\alpha_n \cos(\alpha_n y)(1 - e^{-2\alpha_n a}) = \sum_{n=1}^{\infty} B_n\alpha_n \cos(\alpha_n y)(-1 + e^{-2\alpha_n b}) \tag{9}$$

since a Fourier expansion is unique, we find

$$\lambda A_n(1 - e^{-2\alpha_n a}) + B_n(1 - e^{-2\alpha_n b}) = 0 \tag{10}$$

The second part of Eq. (6) gives

$$\sum_{n=1}^{\infty} B_n\alpha_n \cos(\alpha_n y)(-1 + e^{-2\alpha_n b}) = \frac{1}{\beta}\Big[(\lambda - 1)(1 - y) + \sum B_n \cos(\alpha_n y)(1 + e^{-2\alpha_n b}) - \sum A_n \cos(\alpha_n y)(1 + e^{-2\alpha_n a})\Big] \tag{11}$$

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*Professor, Departments of Mathematics and Mechanical Engineering.

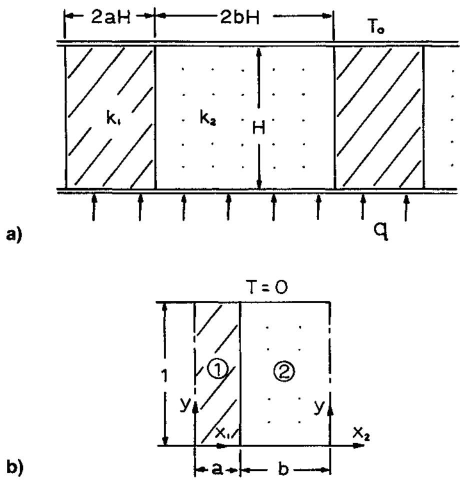

Fig. 1 a) Sandwiched plate and b) normalized coordinates.

multiplying Eq. (11) by $\cos(\alpha_n y)$ and integrating from 0 to 1 yield

$$A_n(1 + e^{-2\alpha_n a}) - B_n[1 + e^{-2\alpha_n b} + \beta\alpha_n(1 - e^{-2\alpha_n b})] = [2(\lambda - 1)/\alpha_n^2] \tag{12}$$

Solving Eqs. (10) and (12) gives

$$A_n = \frac{2(\lambda - 1)(1 - e^{-2\alpha_n b})}{\alpha_n^2 D} \tag{13}$$

$$B_n = \frac{-2\lambda(\lambda - 1)(1 - e^{-2\alpha_n a})}{\alpha_n^2 D} \tag{14}$$

with

$$D \equiv (1 + e^{-2\alpha_n a})(1 - e^{-2\alpha_n b}) + \lambda(1 - e^{-2\alpha_n a})[1 + e^{-2\alpha_n b} + \beta\alpha_n(1 - e^{-2\alpha_n b})] \tag{15}$$

IV. Results and Discussion

The infinite series in Eqs. (7) and (8) are Fourier series in y, whose uniform convergence for piecewise smooth functions is well known. The error is ascertained by increasing the number of terms and noting the difference in the results. For the current problem, convergence is so fast that, in general, only 10 terms are required for a six-figure accuracy.

The effect of thermal contact parameter β is shown in Fig. 2 for $a = 0.5$, $b = 1$, $\lambda = 0.5$. For nonzero β there is a temperature discontinuity across the interface. The difference becomes larger for small y and/or increased β. Thus, thermoelastic stresses would be greatly increased along the interface, contributing further to the thermal and physical separation of the two materials.

The maximum temperature T_m occurs at the axes origin of region 1. Figure 3 shows T_m as a function of various λ, β, a, b. In general, the maximum temperature increases with β, λ, a, and decreases with b. It is most sensitive to width a, low β and λ around 0.1 to 0.2. The local maximum temperature is an important design factor.

In the case where the stringers have hollow cores, a similar analysis can be used, except three or more separate regions are coupled.

Another area of application is that of the thermal properties of laminated composite plates.[2] If the lamina are parallel to

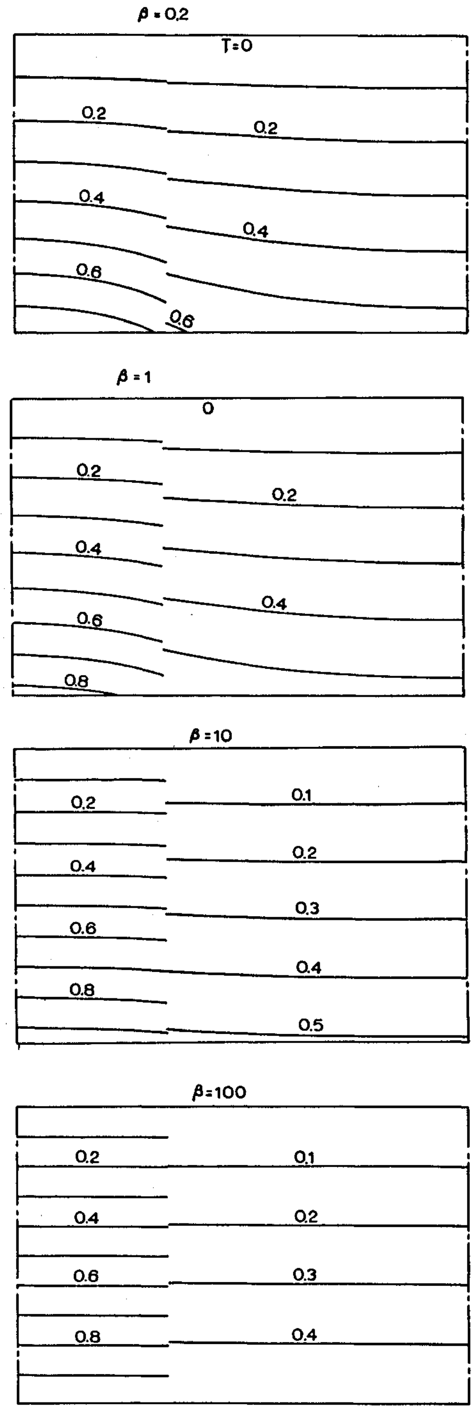

Fig. 2 Isotherms for $a = 0.5$, $b = 1$, $\lambda = 0.5$, and various β.

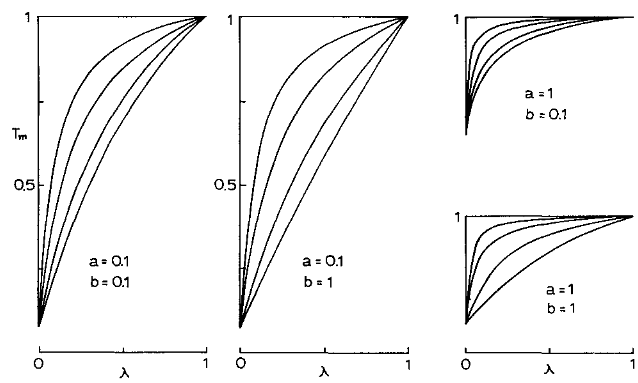

Fig. 3 Maximum temperature curves at constant β. Each set of curves starting from bottom: $\beta = 0, 2, 10, 30$.

the plate surface, the problem is one-dimensional and can be easily solved. The results of the present note can be used if the lamina are perpendicular to the plate surface. In general, both a and b are small for laminated composites.

References

[1]Schneider, P. J., "Conduction," *Handbook of Heat Transfer-Fundamentals*, edited by W. M. Rohsenow, J. P. Hartnett, and E. N. Ganic, 2nd ed., McGraw-Hill, New York, 1985, Chap. 4.

[2]Choo, V. K. S., *Fundamentals of Composite Materials*, Knowen, Dover, DE, 1990, Chap. 5.

Thermal Modeling of Heat Transfer

S. Antony Raj*
Government Arts College, Madras 600 035, India
and
M. Chandrasekar†
Presidency College, Madras 600 005, India

Introduction

GYARMATI'S genuine integral principle based on the fundamentals of the thermodynamics of irreversible processes is formulated for one-dimensional unsteady heat transfer in a semi-infinite solid. The surface temperature variation is restricted to be a power function of time. The integral functional, with the help of the dual field method, is varied with respect to the thermal boundary-layer thickness as the variational parameter. The heat conduction equation is reduced to the Euler-Lagrange equation which is a simple quadratic equation solvable for the dimensionless thermal boundary-layer thickness.

Analysis

We consider a semi-infinite solid $0 \leq x < \infty$ which is initially ($t = 0$) at uniform temperature, and when $t > 0$ the boundary surface at $x = 0$ is kept at a temperature T_0 which is assumed to satisfy a power law of the form

$$T_0(t) = Mt^a \tag{1}$$

The one-dimensional heat conduction equation governing the present system is

$$\rho C_V\left(\frac{\partial T}{\partial t}\right) = \lambda\left(\frac{\partial^2 T}{\partial x^2}\right) \tag{2}$$

where ρ is the density, C_V is the heat capacity at constant volume, and λ is the thermal conductivity. The associated boundary conditions are

$$\begin{aligned} t > 0: \quad x &= 0, \qquad T = T_0(t) \\ x &= \beta(t), \qquad T = 0 \end{aligned} \tag{3}$$

where $\beta(t)$, the hypothetical thermal boundary-layer thickness, is a function of time.

The variational principle which describes in space and time the evolution of irreversible processes has been proposed by Gyarmati[1] in the universal form as

$$\delta \int_V (\sigma - \psi - \Phi)\, \mathrm{d}V = 0 \tag{4}$$

where σ denotes the entropy production, and ψ and Φ are the dissipation potentials. For heat conduction in a rigid body, we write the variational principle (4) in Fourier form[1,2] as

$$\delta \int_V [-\boldsymbol{J}_q \cdot \nabla T - \lambda(\nabla T \cdot \nabla T)/2 - (\boldsymbol{J}_q \cdot \boldsymbol{J}_q)/2\lambda]\, \mathrm{d}V = 0 \tag{5}$$

Here, V denotes a bounded region in three-dimensional space, and T the temperature. The heat flux $\boldsymbol{J}_q$ satisfies the following energy balance equation without source term

$$\rho C_V\left(\frac{\partial T}{\partial t}\right) + \nabla \cdot \boldsymbol{J}_q = 0 \tag{6}$$

The volume integral (5) is maximum at any instant of time[1] for real physical processes, and that maximum is zero. However, in the course of approximation, the volume integral generally becomes a function of time, and therefore, it is natural to integrate it over the time interval $0 < t < \infty$ during which the process has taken place. Hence, we can write the variational principle (5) as below:

$$\delta \int_0^\infty \int_V [-\boldsymbol{J}_q \cdot \nabla T - \lambda(\nabla T)^2/2 - \boldsymbol{J}_q^2/2\lambda]\, \mathrm{d}V\, \mathrm{d}t = 0 \tag{7}$$

In the dual field method[3] we introduce the definition of an approximate temperature field T^* from which the heat flux can be determined using the following constitutive relation:

$$\boldsymbol{J}_q = -\lambda \nabla T^* \tag{8}$$

This method prescribes that T^* satisfies all boundary and smoothing conditions which are satisfied by T. In the case of exact solution, T and T^* are equal. Substituting for $\boldsymbol{J}_q$ in the variational principle, Eq. (7), and in the energy balance, Eq. (6), we get

$$\delta \int_0^\infty \int_V [\nabla T^* \cdot \nabla T - (\nabla T)^2/2 - (\nabla T^*)^2/2]\, \mathrm{d}V\, \mathrm{d}t = 0 \tag{9}$$

$$\rho C_V\left(\frac{\partial T}{\partial t}\right) - \lambda \nabla^2 T^* = 0 \tag{10}$$

In the following we apply the variational principle, Eq. (9), to investigate the present problem of heat transfer in a semi-infinite solid. The variational principle, Eq. (9), equivalent to the one-dimensional heat transfer problem defined by Eq. (2), reduces to

$$\delta \int_0^\infty \int_0^\beta \left[\left(\frac{\partial T}{\partial x}\right)\left(\frac{\partial T^*}{\partial x}\right) - \left(\frac{\partial T}{\partial x}\right)^2 \Big/ 2 - \left(\frac{\partial T^*}{\partial x}\right)^2 \Big/ 2\right] \mathrm{d}x\, \mathrm{d}t = 0 \tag{11}$$

The energy balance, Eq. (10), for the problem considered is

$$\rho C_V\left(\frac{\partial T}{\partial t}\right) - \lambda\left(\frac{\partial^2 T^*}{\partial x^2}\right) = 0 \tag{12}$$

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*Lecturer, Mathematics Department, Government Arts College (Men), Nandanam; currently Lecturer, Mathematics Department, Presidency College (Autonomous), Madras 600 005, India.

†Research Scholar, Mathematics Department, Presidency College (Autonomous).